A priori error estimates of a diffusion equation with Ventcel boundary conditions on curved meshes

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Abstract

In this work is considered a diffusion problem, referred to as the *Ventcel problem*, involving a second order term on the domain boundary (the Laplace-Beltrami operator). A variational formulation of the Ventcel problem is studied, leading to a finite element discretization. The focus is on the construction of high order curved meshes for the discretization of the physical domain and on the definition of the lift operator, which is aimed to transform a function defined on the mesh domain into a function defined on the physical one. This *lift* is defined in a way as to satisfy adapted properties on the boundary, relatively to the trace operator. Error estimations are computed and expressed both in terms of finite element approximation error and of geometrical error, respectively associated to the finite element degree $k \geq 1$ and to the mesh order $r \geq 1$. The numerical experiments we led allow us to validate the results obtained and proved on the *a priori* error estimates depending on the two parameters k and r.

1 Introduction

Motivations The origins of the finite element method (see [1]) can be traced back to the 1950s when engineers started to solve numerically structural mechanics problems in aeronautics. A key point in the analysis of this method is to obtain an estimation of the error produced while approximating the solution u of a problem, typically a PDE, by its finite element approximation u_h. Let us mention that there are two types of error estimation either an *a priori* or an *a posteriori* estimation. The goal of an *a priori* error estimation is to assess the error $\|u - u_h\|$ in terms of the mesh size h, the problem data, and the exact solution u. Conversely, an *a posteriori* estimation depends on h and the computed solution u_h, but not on u. Altogether, the *a priori* error analysis is mainly oriented for theoretical qualification, while the *a posteriori* error analysis serves practical purposes. Together these approaches provide a broad view on the reliability of the approximation method considered. In this work, we focus on *a priori* error estimations.

In various situations, we have to numerically solve a problem, typically a PDE, on lex geometry. This work is aimed at certain industrial applications where the object or material under consideration is surrounded by a thin layer with different properties (typically a surface treatment or a corrosion layer). The presence of this layer causes some difficulties while discretizing

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the domain and numerically solving the problem. To overcome this problem, a classical approach consists in approximating the domain by a similar one without a thin layer but equipped with artificial boundary conditions, like the so-called *Ventcel boundary conditions* [6]. The physical properties of the thin layer are then contained in these new boundary conditions.

These last two topics are going to be the main focus of this paper: we are going to consider the numerical resolution of a (scalar) PDE equipped with higher order boundary conditions, which are the Ventcel boundary conditions, to after that assess the *a priori* error produced by a finite element approximation, on higher order meshes.

The Ventcel problem and its approximation Let Ω be a nonempty bounded connected domain in $\mathbb{R}^d$, $d = 2, 3$, with a smooth boundary $\Gamma := \partial\Omega$. Considering a source term f and a boundary condition g, as well as some given constants $\kappa \geq 0$, α, $\beta > 0$, the Ventcel problem that we will focus on is the following:

$$\left\{ \begin{array}{rcll} -\Delta u + \kappa u & = & f & \text{in } \Omega, \\ -\beta \Delta_\Gamma u + \partial_{\mathrm{n}} u + \alpha u & = & g & \text{on } \Gamma, \end{array} \right. \tag{1}$$

where $\mathbf{n}$ denotes the external unit normal to Γ, $\partial_{\mathrm{n}} u$ the normal derivative of u along Γ and Δ_Γ the Laplace-Beltrami operator.

Notice that the domain Ω is required to be smooth due to the presence of second order boundary conditions. Thus, the physical domain Ω can not be fitted by a polygonal mesh domain. We then resort to high order meshes of geometrical order $r \geq 2$ defined in Section 3, following the work of many authors (see, e.g., [18, 17, 9, 10]). Notice that the domain of the mesh of order r, denoted $\Omega_h^{(r)}$, does not fits the domain Ω, but the numerical results are more accurate as will be exposed in Section 7.

A $\mathbb{P}^k$-Lagrangian finite element method is used with a degree $k \geq 1$ to approximate the exact solution u of System (1) by a finite element function u_h defined on the mesh domain $\Omega_h^{(r)}$. Note that by distinguishing between the parameters r and k, we want to highlight the influence of the geometrical order r of the mesh and of the finite element approximation degree k on the computational error: this allows the degree of the finite element method k to be chosen according to the choice of the geometrical order r. Notice that an *isoparametric approach*, that is taking $k = r$, to this problem is treated in [18, 17].

Since $\Omega_h^{(r)} \neq \Omega$, in order to compare the numerical solution u_h defined on $\Omega_h^{(r)}$ to the exact solution u defined on Ω and to obtain *a priori* error estimations, the notion of *lifting* a function from a domain onto another domain needs to be introduced. The *lift functional* was firstly introduced in the 1970s by many authors (see, e.g., [14, 27, 24, 25]). Among them, let us emphasize the lift based on the orthogonal projection onto the boundary Γ, introduced by Dubois in [14] and further improved in terms of regularity by Elliott *et al.* in [18]. However, the lift defined in [18] does not fit the orthogonal projection on the computational domain's boundary. As will be seen in Section 4.1, this condition is essential to guaranty the theoretical analysis of this problem. In order to address this issue, an alternative definition is introduced in this paper which will be used to perform a numerical study of the computational error of System (1). This modification in the lift definition has a big impact on the error approximation as is observed in the numerical examples in Section 7.

The main result is the following *a priori* error estimations, which will be explained in details and proved in Section 6:

$$\|u - u_h^\ell\|_{\mathrm{L}^2(\Omega,\Gamma)} = O(h^{k+1} + h^{r+1}) \quad \text{and} \quad \|u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} = O(h^k + h^{r+1/2}),$$

where h is the mesh size and u_h^ℓ denotes the *lift* of u_h (given in Definition 5), and $\mathrm{L}^2(\Omega,\Gamma)$ and $\mathrm{H}^1(\Omega,\Gamma)$ are Hilbert spaces defined below.

Paper organization Section 2 contains all the mathematical tools and useful definitions to derive the weak formulation of System (1). Section 3 is devoted to the definition of the high order meshes. In Section 4, are defined the volume and surface lifts, which are the keystones of this work. A Lagrangian finite element space and discrete formulation of System (1) are presented in Section 5, alongside their *lifted forms* onto Ω. The *a priori* error analysis is detailed in Section 6. The paper wraps up in Section 7 with some numerical experiments studying the method convergence rate dependency on the geometrical order r and on the finite element degree k.

2 Notations and needed mathematical tools

Firstly, let us introduce the notations that we adopt in this paper. Throughout this paper, Ω is a nonempty bounded connected open subset of $\mathbb{R}^d$ ($d = 2, 3$) with a smooth (at least $\mathcal{C}^2$) boundary $\Gamma := \partial\Omega$. The unit normal to Γ pointing outwards is denoted by $\mathbf{n}$ and $\partial_{\mathrm{n}} u$ is a normal derivative of a function u. We denote respectively by $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^2(\Gamma)$ the usual Lebesgue spaces endowed with their standard norms on Ω and Γ. Moreover, for $k \geq 1$, $\mathrm{H}^{k+1}(\Omega)$ denotes the usual Sobolev space endowed with its standard norm. We also consider the Sobolev spaces $\mathrm{H}^{k+1}(\Gamma)$ on the boundary as defined e.g. in [23, §2.3]. It is recalled that the norm on $\mathrm{H}^1(\Omega)$ is: $\|u\|^2_{\mathrm{H}^1(\Omega)} := \|u\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla_\Gamma u\|^2_{\mathrm{L}^2(\Omega)}$, where ∇_Γ is the tangential gradient defined below; and that $\|u\|^2_{\mathrm{H}^{k+1}(\Gamma)} := \|u\|^2_{\mathrm{H}^k(\Gamma)} + \|\nabla_\Gamma u\|^2_{\mathrm{H}^k(\Gamma)}$. Throughout this work, we rely on the following Hilbert space (see [23])

$$\mathrm{H}^1(\Omega,\Gamma) := \{u \in \mathrm{H}^1(\Omega),\ u_{|\Gamma} \in \mathrm{H}^1(\Gamma)\},$$

equipped with the norm $\|u\|^2_{\mathrm{H}^1(\Omega,\Gamma)} := \|u\|^2_{\mathrm{H}^1(\Omega)} + \|u\|^2_{\mathrm{H}^1(\Gamma)}$. In a similar way is defined the following space $\mathrm{L}^2(\Omega,\Gamma) := \{u \in \mathrm{L}^2(\Omega),\ u_{|\Gamma} \in \mathrm{L}^2(\Gamma)\}$, equipped with the norm $\|u\|^2_{\mathrm{L}^2(\Omega,\Gamma)} := \|u\|^2_{\mathrm{L}^2(\Omega)} + \|u\|^2_{\mathrm{L}^2(\Gamma)}$. More generally, we define $\mathrm{H}^{k+1}(\Omega,\Gamma) := \{u \in \mathrm{H}^{k+1}(\Omega),\ u_{|\Gamma} \in \mathrm{H}^{k+1}(\Gamma)\}$.

Secondly, we recall the definition of the tangential operators (see, e.g., [22]).

Definition 1. *Let $w \in \mathrm{H}^1(\Gamma)$, $W \in \mathrm{H}^1(\Gamma, \mathbb{R}^d)$ and $u \in \mathrm{H}^2(\Gamma)$. Then the following operators are defined on Γ:*

- *the tangential gradient of w given by $\nabla_\Gamma w := \nabla\tilde{w} - (\nabla\tilde{w}\cdot\mathbf{n})\mathbf{n}$, where $\tilde{w} \in \mathrm{H}^1(\mathbb{R}^d)$ is any extension of w;*
- *the tangential divergence of W given by $\mathrm{div}_\Gamma W := \mathrm{div}\tilde{W} - (\mathrm{D}\tilde{W}\mathbf{n})\cdot\mathbf{n}$, where $\tilde{W} \in \mathrm{H}^1(\mathbb{R}^d,\mathbb{R}^d)$ is any extension of W and $\mathrm{D}\tilde{W} = (\nabla\tilde{W}_i)_{i=1}^d$ is the differential matrix of the extension $\tilde{W}$;*
- *the Laplace-Beltrami operator of u given by $\Delta_\Gamma u := \mathrm{div}_\Gamma(\nabla_\Gamma u)$.*

Additionally, the constructions of the mesh used in Section 3 and of the lift procedure presented in Section 4 are based on the following fundamental result that may be found in [11] and [20, §14.6]. For more details on the geometrical properties of the tubular neighborhood and the orthogonal projection defined below, we refer to [12, 13, 16].

Proposition 1. *Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^d$ with a $\mathcal{C}^2$ boundary $\Gamma = \partial\Omega$. Let $\mathrm{d} : \mathbb{R}^d \to \mathbb{R}$ be the signed distance function with respect to Γ defined by,*

$$\mathrm{d}(x) := \begin{cases} -\mathrm{dist}(x,\Gamma) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Gamma, \\ \mathrm{dist}(x,\Gamma) & \text{otherwise,} \end{cases} \qquad \text{with} \quad \mathrm{dist}(x,\Gamma) := \inf\{|x-y|, \quad y \in \Gamma\}.$$

Then there exists a tubular neighborhood $\mathcal{U}_\Gamma := \{x \in \mathbb{R}^d; |\mathrm{d}(x)| < \delta_\Gamma\}$ of Γ, of sufficiently small width δ_Γ, where d is a $\mathcal{C}^2$ function. Its gradient $\nabla\mathrm{d}$ is an extension of the external unit normal $\mathbf{n}$ to Γ. Additionally, in this neighborhood $\mathcal{U}_\Gamma$, the orthogonal projection b onto Γ is uniquely defined and given by,

$$b : \ x \in \mathcal{U}_\Gamma \longmapsto b(x) := x - \mathrm{d}(x)\nabla\mathrm{d}(x) \in \Gamma.$$

Finally, the variational formulation of Problem (1) is obtained, using the integration by parts formula on the surface Γ (see, e.g. [22]), and is given by,

$$\text{find } u \in \mathrm{H}^1(\Omega,\Gamma) \text{ such that } a(u,v) = l(v), \ \forall \ v \in \mathrm{H}^1(\Omega,\Gamma), \tag{2}$$

where the bilinear form a, defined on $\mathrm{H}^1(\Omega,\Gamma)^2$, is given by,

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \kappa \int_\Omega uv \,\mathrm{d}x + \beta \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \,\mathrm{d}\sigma + \alpha \int_\Gamma uv \,\mathrm{d}\sigma,$$

and the linear form l, defined on $\mathrm{H}^1(\Omega,\Gamma)$, is given by,

$$l(v) := \int_\Omega fv \,\mathrm{d}x + \int_\Gamma gv \,\mathrm{d}\sigma.$$

The following theorem claims the well-posedness of the problem (2) proven in [8, th. 2] and [23, th. 3.3] and establishes the solution regularity proven in [23, th. 3.4].

Theorem 1. *Let Ω and $\Gamma = \partial\Omega$ be as stated previously. Let α, $\beta > 0$, $\kappa \geq 0$, and $f \in \mathrm{L}^2(\Omega)$, $g \in \mathrm{L}^2(\Gamma)$. Then there exists a unique solution $u \in \mathrm{H}^1(\Omega,\Gamma)$ to problem (2).*

Moreover, if Γ is of class $\mathcal{C}^{k+1}$, and $f \in \mathrm{H}^{k-1}(\Omega)$, $g \in \mathrm{H}^{k-1}(\Gamma)$, then the solution u of (2) is in $\mathrm{H}^{k+1}(\Omega,\Gamma)$ and is the strong solution of the Ventcel problem (1). Additionally, there exists $c > 0$ such that the following inequality holds,

$$\|u\|_{\mathrm{H}^{k+1}(\Omega,\Gamma)} \leq c(\|f\|_{\mathrm{H}^{k-1}(\Omega)} + \|g\|_{\mathrm{H}^{k-1}(\Gamma)}).$$

3 Curved mesh definition

In this section we briefly recall the construction of curved meshes of geometrical order $r \geq 1$ of the domain Ω and introduce some notations. We refer to [8, Section 2] for details and examples (see also [18, 27, 14, 2]). Recall for $r \geq 1$, the set of polynomials in $\mathbb{R}^d$ of order r or less is denoted by $\mathbb{P}^r$. From now on, the domain Ω, is assumed to be at least $\mathcal{C}^{r+2}$ regular, and $\hat{T}$ denotes the reference simplex of dimension d. In a nutshell, the way to proceed is the following.

1. Construct an affine mesh $\mathcal{T}_h^{(1)}$ of Ω composed of simplices T and define the affine transformation $F_T : \ \hat{T} \to T := F_T(\hat{T})$ associated to each simplice T.

2. For each simplex $T \in \mathcal{T}_h^{(1)}$, a mapping $F_T^{(e)} : \ \hat{T} \to T^{(e)} := F_T^{(e)}(\hat{T})$ is designed and the resulting *exact elements* $T^{(e)}$ will form a curved exact mesh $\mathcal{T}_h^{(e)}$ of Ω.

3. For each $T \in \mathcal{T}_h^{(1)}$, the mapping $F_T^{(r)}$ is the $\mathbb{P}^r$ interpolant of $F_T^{(e)}$. The curved mesh $\mathcal{T}_h^{(r)}$ of order r is composed of the elements $T^{(r)} := F_T^{(r)}(\hat{T})$.

3.1 Affine mesh $\mathcal{T}_h^{(1)}$

Let $\mathcal{T}_h^{(1)}$ be a polyhedral mesh of Ω made of simplices of dimension d (triangles or tetrahedra), it is chosen as quasi-uniform and henceforth shape-regular (see [7, definition 4.4.13]). Define the mesh size $h := \max\{\text{diam}(T);\ T \in \mathcal{T}_h^{(1)}\}$, where diam$(T)$ is the diameter of T. The mesh domain is denoted by $\Omega_h^{(1)} := \cup_{T\in\mathcal{T}_h^{(1)}} T$. Its boundary denoted by $\Gamma_h^{(1)} := \partial\Omega_h^{(1)}$ is composed of $(d-1)$-dimensional simplices that form a mesh of $\Gamma = \partial\Omega$. The vertices of $\Gamma_h^{(1)}$ are assumed to lie on Γ.

For $T \in \mathcal{T}_h^{(1)}$, we define an affine function that maps the reference element onto T,

$$F_T : \hat{T} \to T := F_T(\hat{T})$$

Remark 1. *For a sufficiently small mesh size h, the mesh boundary satisfies $\Gamma_h^{(1)} \subset \mathcal{U}_\Gamma$, where $\mathcal{U}_\Gamma$ is the tubular neighborhood given in proposition 1. This guaranties that the orthogonal projection $b : \Gamma_h^{(1)} \to \Gamma$ is one to one which is required for the construction of the exact mesh.*

3.2 Exact mesh $\mathcal{T}_h^{(e)}$

In the 1970's, Scott gave an explicit construction of an exact triangulation in two dimensions in [27], generalised by Lenoir in [24] afterwards (see also [18, §4] and [17, §3.2]). The present definition of an exact transformation $F_T^{(e)}$ combines the definitions found in [24, 27] with the projection b as used in [14].

Let us first point out that for a sufficiently small mesh size h, a mesh element T cannot have $d+1$ vertices on the boundary Γ, due to the quasi uniform assumption imposed on the mesh $\mathcal{T}_h^{(1)}$. A mesh element is said to be an internal element if it has at most one vertex on the boundary Γ.

Definition 2. *Let $T \in \mathcal{T}_h^{(1)}$ be a non-internal element (having at least 2 vertices on the boundary). Denote $v_i = F_T(\hat{v}_i)$ as its vertices, where $\hat{v}_i$ are the vertices of $\hat{T}$. We define $\varepsilon_i = 1$ if $v_i \in \Gamma$ and $\varepsilon_i = 0$ otherwise. To $\hat{x} \in \hat{T}$ is associated its barycentric coordinates λ_i associated to the vertices $\hat{v}_i$ of $\hat{T}$ and $\lambda^*(\hat{x}) := \sum_{i=1}^{d+1} \varepsilon_i \lambda_i$ (shortly denoted by λ^*). Finally, we define $\hat{\sigma} := \{\hat{x} \in \hat{T}; \lambda^*(\hat{x}) = 0\}$ and the function $\hat{y} := \dfrac{1}{\lambda^*}\sum_{i=1}^{d+1} \varepsilon_i \lambda_i \hat{v}_i \in \hat{T}$, which is well defined on $\hat{T}\backslash\hat{\sigma}$.*

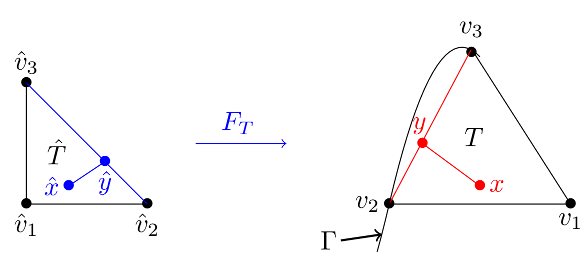

Figure 1: Visualisation of the two functions $\hat{y} : \hat{T} \mapsto \hat{T}$ and $y : T \mapsto \partial T \cap \Gamma$ in definition 3 in a 2D case

Consider a non-internal mesh element $T \in \mathcal{T}_h^{(1)}$ and the affine transformation F_T. In the two dimensional case, $F_T(\hat{\sigma})$ will consist of the only vertex of T that is not on the boundary Γ.

In the three dimensional case, the tetrahedral T either has 2 or 3 vertices on the boundary. In the first case, $F_T(\hat{\sigma})$ is the edge of T joining its two internal vertices. In the second case, $F_T(\hat{\sigma})$ is the only vertex of T.

Definition 3. *We denote $\mathcal{T}_h^{(e)}$ the mesh consisting of all exact elements $T^{(e)} = F_T^{(e)}(\hat{T})$, where $F_T^{(e)} = F_T$ for all internal elements, as for the case of non-internal elements $F_T^{(e)}$ is given by,*

$$\begin{array}{rccl} F_T^{(e)}: & \hat{T} & \longrightarrow & T^{(e)} := F_T^{(e)}(\hat{T}) \\ & \hat{x} & \longmapsto & F_T^{(e)}(\hat{x}) := \begin{cases} x & \text{if } \hat{x} \in \hat{\sigma}, \\ x + (\lambda^*)^{r+2}(b(y) - y) & \text{if } \hat{x} \in \hat{T} \backslash \hat{\sigma}, \end{cases} \end{array} \tag{3}$$

with $x = F_T(\hat{x})$ and $y = F_T(\hat{y})$. It has been proven in [18] that $F_T^{(e)}$ is a $\mathcal{C}^1$-diffeomorphism and C^{r+1} regular on $\hat{T}$.

Remark 2. *For $x \in T \cap \Gamma_h$, we have that $\lambda^* = 1$ and so $y = x$ inducing that $F_T^{(e)}(\hat{x}) = b(x)$. Then $F_T^{(e)} \circ F_T^{-1} = b$ on $T \cap \Gamma_h$.*

3.3 Curved mesh $\mathcal{T}_h^{(r)}$ of order r

The exact mapping $F_T^{(e)}$, defined in (3), is interpolated as a polynomial of order $r \geq 1$ in the classical $\mathbb{P}^r$-Lagrange basis on $\hat{T}$. The interpolant is denoted by $F_T^{(r)}$, which is a $\mathcal{C}^1$-diffeomorphism and is in $\mathcal{C}^{r+1}(\hat{T})$ (see [9, chap. 4.3]). For more exhaustive details and properties of this transformation, we refer to [18, 10, 9]. Note that, by definition, $F_T^{(r)}$ and $F_T^{(e)}$ coincide on all $\mathbb{P}^r$-Lagrange nodes. The curved mesh of order r is $\mathcal{T}_h^{(r)} := \{T^{(r)}; T \in \mathcal{T}_h^{(1)}\}$, $\Omega_h^{(r)} := \cup_{T^{(r)} \in \mathcal{T}_h^{(r)}} T^{(r)}$ is the mesh domain and $\Gamma_h^{(r)} := \partial \Omega_h^{(r)}$ is its boundary.

4 Functional lift

We recall that $r \geq 1$ is the geometrical order of the curved mesh. With the help of aforementioned transformations, we define *lifts* to transform a function on a domain $\Omega_h^{(r)}$ or $\Gamma_h^{(r)}$ into a function defined on Ω or Γ respectively, in order to compare the numerical solutions to the exact one.

We recall that the idea of lifting a function from the discrete domain onto the continuous one was already treated and discussed in many articles dating back to the 1970's, like [25, 27, 24, 2] and others. Surface lifts were firstly introduced in 1988 by Dziuk in [15], to the extend of our knowledge, and discussed in more details and applications by Demlow in many of his articles (see [12, 13, 3, 5]).

4.1 Surface and volume lift definitions

Definition 4 (Surface lift)**.** *Let $u_h \in \mathrm{L}^2(\Gamma_h^{(r)})$. The surface lift $u_h^L \in \mathrm{L}^2(\Gamma)$ associated to u_h is defined by,*

$$u_h^L \circ b := u_h,$$

where $b : \Gamma_h^{(r)} \to \Gamma$ is the orthogonal projection, defined in Proposition 1. Likewise, to $u \in \mathrm{L}^2(\Gamma)$ is associated its inverse lift u^{-L} given by, $u^{-L} := u \circ b \in \mathrm{L}^2(\Gamma_h^{(r)})$.

The use of the orthogonal projection b to define the surface lift is natural since b is well defined on the tubular neighborhood $\mathcal{U}_\Gamma$ of Γ (see Proposition 1) and henceforth on $\Gamma_h^{(r)} \subset \mathcal{U}_\Gamma$ for sufficiently small mesh size h.

A volume lift is defined, using the notations in definition 2, we introduce the transformation $G_h^{(r)} : \Omega_h^{(r)} \to \Omega$ (see figure 2) given piecewise for all $T^{(r)} \in \mathcal{T}_h^{(r)}$ by,

$$G_h^{(r)}{}_{|_{T^{(r)}}} := F_{T^{(r)}}^{(e)} \circ (F_T^{(r)})^{-1}, \; F_{T^{(r)}}^{(e)}(\hat{x}) := \begin{cases} x & \text{if } \hat{x} \in \hat{\sigma} \\ x + (\lambda^*)^{r+2}(b(y) - y) & \text{if } \hat{x} \in \hat{T} \backslash \hat{\sigma} \end{cases}, \tag{4}$$

with $x := F_T^{(r)}(\hat{x})$ and $y := F_T^{(r)}(\hat{y})$ (see figure 1 for the affine case). Notice that this implies that $G_h^{(r)}{}_{|_{T^{(r)}}} = id_{|_{T^{(r)}}}$, for any internal mesh element $T^{(r)} \in \mathcal{T}_h^{(r)}$. Note that, by construction, $G_h^{(r)}$ is globally continuous and piecewise differentiable on each mesh element. For the remainder of this article, the following notations are crucial. $\mathrm{D}G_h^{(r)}$ denotes the differential of $G_h^{(r)}$, $(\mathrm{D}G_h^{(r)})^t$ is its transpose and J_h is its Jacobin.

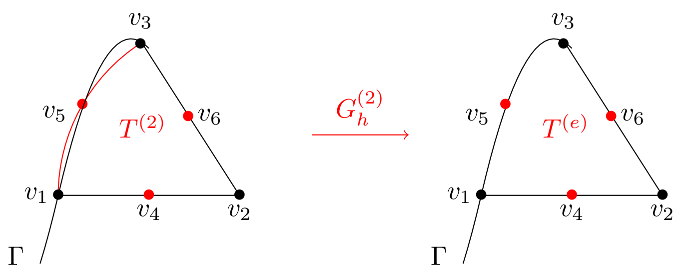

Figure 2: Visualisation of $G_h^{(2)} : T^{(2)} \to T^{(e)}$ in a 2D case, for a quadratic case $r = 2$.

Definition 5 (Volume lift)**.** *Let $u_h \in \mathrm{L}^2(\Omega_h^{(r)})$. We define the volume lift associated to u_h, denoted $u_h^\ell \in \mathrm{L}^2(\Omega)$, by,*

$$u_h^\ell \circ G_h^{(r)} := u_h.$$

In a similar way, to $u \in \mathrm{L}^2(\Omega)$ is associated its inverse lift $u^{-\ell} \in \mathrm{L}^2(\Omega_h^{(r)})$ given by $u^{-\ell} := u \circ G_h^{(r)}$.

Proposition 2. *The volume and surface lifts coincide on $\Gamma_h^{(r)}$,*

$$\forall \; u_h \in \mathrm{H}^1(\Omega_h^{(r)}), \quad (\mathrm{Tr}\; u_h)^L = \mathrm{Tr}(u_h^\ell).$$

Consequently, the surface lift v_h^L (resp. the inverse lift v^{-L}) will now be simply denoted by v_h^ℓ (resp. $v^{-\ell}$).

Proof. Taking $x \in T^{(r)} \cap \Gamma_h^{(r)}$, $\hat{x} = (F_T^{(r)})^{(-1)}(x)$ satisfies $\lambda^* = 1$ and so $\hat{y} = \hat{x}$ and $y = x$. Thus $F_{T^{(r)}}^{(e)}(\hat{x}) = b(x)$, in other words,

$$G_h^{(r)}(x) = F_{T^{(r)}}^{(e)} \circ (F_T^{(r)})^{-1}(x) = b(x), \qquad \forall \; x \in T^{(r)} \cap \Gamma_h^{(r)}.$$

□

Proposition 3. *Let $T^{(r)} \in \mathcal{T}_h^{(r)}$. Then the mapping ${G_h^{(r)}}_{|_{T^{(r)}}}$ is $\mathcal{C}^{r+1}(T^{(r)})$ regular and a $\mathcal{C}^1$-diffeomorphism from $T^{(r)}$ onto $T^{(e)}$. Additionally, for a sufficiently small mesh size h, there exists a constant $c > 0$, independent of h, such that,*

$$\forall\, x \in T^{(r)}, \qquad \|\mathrm{D}G_h^{(r)}(x) - \mathrm{Id}\| \leq ch^r \qquad \text{and} \qquad |J_h(x) - 1| \leq ch^r, \tag{5}$$

where $G_h^{(r)}$ is defined in (4) and J_h is its Jacobin.

The full proof of this proposition is partially adapted from [18] and has been detailed in appendix A.

Remark 3 (Lift regularity)**.** *The lift transformation $G_h^{(r)} : \Omega_h^{(r)} \to \Omega$ in (4) involves the function,*

$$\rho_{T^{(r)}} : \ \hat{x} \in \hat{T} \mapsto (\lambda^*)^s (b(y) - y),$$

with an exponent $s = r + 2$ inherited from [18]: this exponent value guaranties the $\mathcal{C}^{r+1}$ (piecewise) regularity of the function $G_h^{(r)}$. However, decreasing that value to $s = 2$ still ensures that $G_h^{(r)}$ is a (piecewise) $\mathcal{C}^1$ diffeomorphism and also that Inequalities (5) hold: this can be seen when examining the proof of Proposition 3 in Appendix A. Consequently, the convergence theorem 2 still holds when setting $s = 2$ in the definition of $\rho_{T^{(r)}}$.

Remark 4 (Former lift definition)**.** *The volume lift defined in (5) is an adaptation of the lift definition in [18], which however does not fulfill the property 2. Precisely, in [18], to $u_h \in \mathrm{H}^1(\Omega_h^{(r)})$ is associated the lifted function $u_h^\ell \in \mathrm{H}^1(\Omega)$, given by $u_h^\ell \circ G_h := u_h$, where $G_h : \Omega_h^{(r)} \to \Omega$ is defined piecewise, for each mesh element $T^{(r)} \in \mathcal{T}_h^{(r)}$, by ${G_h}_{|_{T^{(r)}}} := F_T^{(e)} \circ (F_T^{(r)})^{-1}$, where T is the affine element relative to $T^{(r)}$, $F_T^{(e)}$ is defined in (3) and $F_T^{(r)}$ is its $\mathbb{P}^r$-Lagrangien interpolation given in section 3.3. However, this transformation does not coincide with the orthogonal projection b, on the mesh boundary $\Gamma_h^{(r)}$. Indeed, since $F_T^{(e)} \circ F_T^{-1} = b$ on $T \cap \Gamma_h$ (see Remark 2), we have,*

$$G_h(x) = b \circ F_T \circ (F_T^{(r)})^{-1}(x) \neq b(x), \quad \forall\, x \in \Gamma_h^{(r)} \cap T^{(r)}.$$

Consequently in this case, $(\mathrm{Tr}\ u_h)^L \neq \mathrm{Tr}(u_h^\ell)$.

4.2 Lift of the variational formulation

With the lift operator, one may express an integral over $\Gamma_h^{(r)}$ (resp. $\Omega_h^{(r)}$) with respect to one over Γ (resp. Ω), as will be discussed in this section.

Surface integrals In this subsection, all results stated may be found alongside their proofs in [12, 4], but we recall some necessary informations for the sake of completeness. For extensive details, we also refer to [13, 16, 15]. Throughout the rest of the paper, $\mathrm{d}\sigma$ and $\mathrm{d}\sigma_h$ denote respectively the surface measures on Γ and on $\Gamma_h^{(r)}$.

Let J_b be the Jacobian of the orthogonal projection b, defined in Proposition 1, such that $\mathrm{d}\sigma(b(x)) = J_b(x)\,\mathrm{d}\sigma_h(x)$, for all $x \in \Gamma_h^{(r)}$. Notice that J_b is bounded independently of h and its detailed expression may be found in [12, 13]. Consider also the lift of J_b given by $J_b^\ell \circ b = J_b$ (see Definition 4).

Let $u_h, v_h \in \mathrm{H}^1(\Gamma_h)$ with $u_h^\ell, v_h^\ell \in \mathrm{H}^1(\Gamma)$ as their respected lifts. Then, one has,

$$\int_{\Gamma_h^{(r)}} u_h v_h \,\mathrm{d}\sigma_h = \int_\Gamma u_h^\ell v_h^\ell \frac{\mathrm{d}\sigma}{J_b^\ell}. \tag{6}$$

A similar equation may be written with tangential gradients. We start by given the following notations. The outer unit normal vectors over Γ and $\Gamma_h^{(r)}$ are respectively denoted by $\mathbf{n}$ and $\mathbf{n}_{hr}$. Denote $P := \mathrm{Id} - \mathbf{n} \otimes \mathbf{n}$ and $P_h := \mathrm{Id} - \mathbf{n}_{hr} \otimes \mathbf{n}_{hr}$ respectively as the orthogonal projections over the tangential spaces of Γ and $\Gamma_h^{(r)}$. Additionally, the Weingarten map $\mathcal{H} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is given by $\mathcal{H} := \mathrm{D}^2\mathrm{d}$, where d is the signed distance function (see Proposition 1). With the previous notations, we have,

$$\nabla_{\Gamma_h} v_h(x) = P_h(I - \mathrm{d}\mathcal{H})P\nabla_\Gamma v_h^\ell(b(x)), \qquad \forall\, x \in \Gamma_h^{(r)}.$$

Using this equality, we may derive the following expression,

$$\int_{\Gamma_h^{(r)}} \nabla_{\Gamma_h^{(r)}} u_h \cdot \nabla_{\Gamma_h^{(r)}} v_h \, \mathrm{d}\sigma_h = \int_\Gamma A_h^\ell \nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma v_h^\ell \, \mathrm{d}\sigma, \tag{7}$$

where A_h^ℓ is the lift of the matrix A_h given by,

$$A_h(x) := \frac{1}{J_b(x)} P(I - \mathrm{d}\mathcal{H})P_h(I - \mathrm{d}\mathcal{H})P(x), \qquad \forall\, x \in \Gamma_h^{(r)}. \tag{8}$$

Volume integrals Similarly, consider $u_h, v_h \in \mathrm{H}^1(\Omega_h)$ and let $u_h^\ell, v_h^\ell \in \mathrm{H}^1(\Omega)$ be their respected lifts (see Definition 5), we have,

$$\int_{\Omega_h} u_h v_h \, \mathrm{d}x = \int_\Omega u_h^\ell v_h^\ell \frac{1}{J_h^\ell} dy, \tag{9}$$

where J_h denotes the Jacobian of $G_h^{(r)}$ and J_h^ℓ is its lift given by $J_h^\ell \circ G_h^{(r)} = J_h$.

Additionally, the gradient can be written as follows, for any $x \in \Omega_h^{(r)}$,

$$\nabla v_h(x) = \nabla(v_h^\ell \circ G_h^{(r)})(x) = {}^\mathsf{T}\mathrm{D}G_h^{(r)}(x)(\nabla v_h^\ell) \circ (G_h^{(r)}(x)).$$

Using a change of variables $z = G_h^{(r)}(x) \in \Omega$, one has, $(\nabla v_h)^\ell(z) = {}^\mathsf{T}\mathrm{D}G_h^{(r)}(x)\nabla v_h^\ell(z)$. Finally, introducing the notation,

$$\mathcal{G}_h^{(r)}(z) := {}^\mathsf{T}\mathrm{D}G_h^{(r)}(x), \tag{10}$$

one has,

$$\int_{\Omega_h^{(r)}} \nabla u_h \cdot \nabla v_h \, \mathrm{d}x = \int_\Omega \mathcal{G}_h^{(r)}(\nabla u_h^\ell) \cdot \mathcal{G}_h^{(r)}(\nabla v_h^\ell) \frac{\mathrm{d}x}{J_h^\ell}. \tag{11}$$

4.3 Useful estimations

Surface estimations We recall two important estimates proved in [12]. There exists a constant $c > 0$ independent of h such that,

$$||A_h^\ell - P||_{\mathrm{L}^\infty(\Gamma)} \le ch^{r+1} \qquad \text{and} \qquad \left\| 1 - \frac{1}{J_b^\ell} \right\|_{\mathrm{L}^\infty(\Gamma)} \le ch^{r+1}, \tag{12}$$

where A_h^ℓ is the lift of A_h defined in (8) and J_b is the Jacobin of the projection b.

Volume estimations A direct consequence of the proposition 3 is that both $\mathrm{D}G_h^{(r)}$ and J_h are bounded on every $T^{(r)} \in \mathcal{T}_h^{(r)}$. As an extension of that, by Definition 5 of the lift, both $\mathcal{G}_h^{(r)}$ and J_h^ℓ are also bounded on $T^{(e)}$. Additionally, the inequalities (5) will not be directly used in the error estimations in Section 6, the following inequalities will be used instead,

$$\forall\, x \in T^{(e)}, \quad \|\mathcal{G}_h^{(r)}(x) - \mathrm{Id}\| \leq ch^r \qquad \text{and} \qquad \left|\frac{1}{J_h^\ell(x)} - 1\right| \leq ch^r, \tag{13}$$

where $\mathcal{G}_h^{(r)}$ is given in (10). These inequalities are a consequence of the lift applied on the inequalities (5).

Remark 5. *Let us emphasize that, there exists an equivalence between the* H^m*-norms over* Ω_h *(resp.* Γ_h*) and the* H^m*-norms over* Ω *(resp.* Γ*), for* $m = 0, 1$*. Let* $v_h \in \mathrm{H}^1(\Omega_h, \Gamma_h)$ *and let* $v_h^\ell \in \mathrm{H}^1(\Omega, \Gamma)$ *be its lift, then for* $m = 0, 1$*, there exist strictly positive constants independent of* h *such that,*

$$\begin{array}{rclcl} c_1\|v_h^\ell\|_{\mathrm{H}^m(\Omega)} & \leq & \|v_h\|_{\mathrm{H}^m(\Omega_h)} & \leq & c_2\|v_h^\ell\|_{\mathrm{H}^m(\Omega)}, \\ c_3\|v_h^\ell\|_{\mathrm{H}^m(\Gamma)} & \leq & \|v_h\|_{\mathrm{H}^m(\Gamma_h)} & \leq & c_4\|v_h^\ell\|_{\mathrm{H}^m(\Gamma)}. \end{array}$$

The second estimations are proved in [12]. As for the first inequalities, one may prove them while using the equations (9) *and* (11)*. They hold due to the fact that* J_h *and* $\mathrm{D}G_h^{(r)}$ *(respectively* $\frac{1}{J_h^\ell}$ *and* $\mathcal{G}_h^{(r)}$*) are bounded on* $T^{(r)}$ *(resp.* $T^{(e)}$*), as a consequence of the proposition 3 and the inequalities in* (13)*.*

5 Finite element approximation

In this section, is presented the finite element approximation of problem (1) using $\mathbb{P}^k$-Lagrange finite element approximation. We refer to [19, 9] for more details on finite element methods.

5.1 Finite element spaces and interpolant definition

Let $k \geq 1$, given a curved mesh $\mathcal{T}_h^{(r)}$, the $\mathbb{P}^k$-Lagrangian finite element space is given by,

$$\mathbb{V}_h := \{\chi \in C^0(\Omega_h^{(r)});\ \chi_{|T} = \hat{\chi} \circ (F_T^{(r)})^{-1},\ \hat{\chi} \in \mathbb{P}^k(\hat{T}),\ \forall\, T \in \mathcal{T}_h^{(r)}\}.$$

Let the $\mathbb{P}^r$-Lagrangian interpolation operator be denoted by $\mathcal{I}^{(r)} : v \in \mathcal{C}^0(\Omega_h^{(r)}) \mapsto \mathcal{I}^{(r)}(v) \in \mathbb{V}_h$. The lifted finite element space (see Section 4.1 for the lift definition), is defined by,

$$\mathbb{V}_h^\ell := \{v_h^\ell;\ v_h \in \mathbb{V}_h\},$$

and its lifted interpolation operator $\mathcal{I}^\ell$ given by,

$$\begin{array}{rccl} \mathcal{I}^\ell: & \mathcal{C}^0(\Omega) & \longrightarrow & \mathbb{V}_h^\ell \\ & v & \longmapsto & \mathcal{I}^\ell(v) := \big(\mathcal{I}^{(r)}(v^{-\ell})\big)^\ell. \end{array} \tag{14}$$

Notice that, since Ω is an open subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, then we have the following Sobolev injection $\mathrm{H}^{k+1}(\Omega) \hookrightarrow \mathcal{C}^0(\Omega)$. Thus, any function $w \in \mathrm{H}^{k+1}(\Omega)$ may be associated to an interpolation element $\mathcal{I}^\ell(w) \in \mathbb{V}_h^\ell$.

The lifted interpolation operator plays a part in the error estimation and the following interpolation inequality will display the finite element error in the estimations.

Proposition 4. *Let $v \in \mathrm{H}^{k+1}(\Omega,\Gamma)$ and $2 \le m \le k+1$. There exists a constant $c > 0$ independent of h such that the interpolation operator $\mathcal{I}^\ell$ satisfies the following inequality,*

$$\|v - \mathcal{I}^\ell v\|_{\mathrm{L}^2(\Omega,\Gamma)} + h\|v - \mathcal{I}^\ell v\|_{\mathrm{H}^1(\Omega,\Gamma)} \le c h^m \|v\|_{\mathrm{H}^m(\Omega,\Gamma)}.$$

Proof. Using the norm equivalence in Remark 5, this inequality derives from given interpolation theory (see [2, Corollary 4.1] for norms over Ω and [12, 13] for norms over Γ). Indeed, for $v \in \mathrm{H}^{k+1}(\Omega,\Gamma)$ and $s = 0, 1$, we have,

$$\begin{aligned}\|v - \mathcal{I}^\ell v\|_{\mathrm{H}^s(\Omega,\Gamma)} &= \left\|(v^{-\ell})^\ell - \left(\mathcal{I}^{(r)}(v^{-\ell})\right)^\ell\right\|_{\mathrm{H}^s(\Omega,\Gamma)} \le c\left\|v^{-\ell} - \mathcal{I}^{(r)}(v^{-\ell})\right\|_{\mathrm{H}^s(\Omega_h^{(r)},\Gamma_h^{(r)})} \\ &\le c h^{m-s}\|v^{-\ell}\|_{\mathrm{H}^m(\Omega_h^{(r)},\Gamma_h^{(r)})} \le c h^{m-s}\|v\|_{\mathrm{H}^m(\Omega,\Gamma)},\end{aligned}$$

for a constant $c > 0$ independent of h. □

5.2 Finite element formulation

From now on, to simplify the notations, we denote Ω_h and Γ_h to refer to $\Omega_h^{(r)}$ and $\Gamma_h^{(r)}$, for any geometrical order $r \ge 1$.

Discrete formulation Given $f \in \mathrm{L}^2(\Omega)$ and $g \in \mathrm{L}^2(\Gamma)$ the right hand side of Problem (1), we define (following [18, 12]) the following linear form l_h on $\mathbb{V}_h$ by,

$$l_h(v_h) := \int_{\Omega_h} v_h f^{-\ell} J_h \,\mathrm{d}x + \int_{\Gamma_h} v_h g^{-\ell} J_b \,\mathrm{d}\sigma_h,$$

where J_h (resp. J_b) is the Jacobin of $G_h^{(r)}$ (resp. the orthogonal projection b). With this definition, $l_h(v_h) = l(v_h^\ell)$, for any $v_h \in \mathbb{V}_h$, where l is the right hand side in the formulation (2).

The approximation problem is to find $u_h \in \mathbb{V}_h$ such that,

$$a_h(u_h, v_h) = l_h(v_h), \qquad \forall\, v_h \in \mathbb{V}_h, \tag{15}$$

where a_h is the following bilinear form, defined on $\mathbb{V}_h \times \mathbb{V}_h$,

$$\begin{aligned} a_h(u_h, v_h) := &\int_{\Omega_h} \nabla u_h \cdot \nabla v_h \,\mathrm{d}x + \kappa \int_{\Omega_h} u_h v_h \,\mathrm{d}x \\ &+ \beta \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h \,\mathrm{d}\sigma_h + \alpha \int_{\Gamma_h} u_h v_h \,\mathrm{d}\sigma_h,\end{aligned}$$

Remark 6. *Since a_h is bilinear symmetric positively defined on a finite dimensional space, then there exists a unique solution $u_h \in \mathbb{V}_h$ to the discrete problem* (15).

Lifted discrete formulation We define the lifted bilinear form a_h^ℓ, defined on $\mathbb{V}_h^\ell \times \mathbb{V}_h^\ell$, throughout,

$$a_h^\ell(u_h^\ell, v_h^\ell) = a_h(u_h, v_h) \quad \text{ for } u_h, v_h \in \mathbb{V}_h,$$

applying (11), (9), (7) and (6), its expression is given by,

$$\begin{aligned} a_h^\ell(u_h^\ell, v_h^\ell) = &\int_\Omega \mathcal{G}_h^{(r)}(\nabla u_h^\ell) \cdot \mathcal{G}_h^{(r)}(\nabla v_h^\ell) \frac{\mathrm{d}x}{J_h^\ell} + \beta \int_\Gamma A_h^\ell \nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma v_h^\ell \,\mathrm{d}\sigma \\ &+ \kappa \int_\Omega (u_h)^\ell (v_h)^\ell \frac{\mathrm{d}x}{J_h^\ell} + \alpha \int_\Gamma (u_h)^\ell (v_h)^\ell \frac{\mathrm{d}\sigma}{J_b^\ell}.\end{aligned}$$

Keeping in mind that u is the solution of (2) and u_h^ℓ is the lift of the solution of (15), for any $v_h^\ell \in \mathbb{V}_h^\ell \subset \mathrm{H}^1(\Omega, \Gamma)$, we notice that,

$$a(u, v_h^\ell) = l(v_h^\ell) = l_h(v_h) = a_h(u_h, v_h) = a_h^\ell(u_h^\ell, v_h^\ell). \tag{16}$$

Using the previous points, we can also define the lifted formulation of the discrete problem (15) by: find $u_h^\ell \in \mathbb{V}_h^\ell$ such that,

$$a_h^\ell(u_h^\ell, v_h^\ell) = l(v_h^\ell), \qquad \forall\ v_h^\ell\ \in \mathbb{V}_h^\ell.$$

6 Error analysis

Throughout this section, c refers to a positive constant independent of the mesh size h. From now on, the domain Ω, is assumed to be at least $\mathcal{C}^{k+1}$ regular, and the source terms in problem (1) are assumed more regular: $f \in \mathrm{H}^{k-1}(\Omega)$ and $g \in \mathrm{H}^{k-1}(\Gamma)$. Then according to [23, Theorem 3.4], the exact solution u of Problem (1) is in $\mathrm{H}^{k+1}(\Omega, \Gamma)$.

Our goal in this section is to prove the following theorem.

Theorem 2. *Let $u \in \mathrm{H}^{k+1}(\Omega, \Gamma)$ be the solution of the variational problem* (2) *and $u_h \in \mathbb{V}_h$ be the solution of the finite element formulation* (15)*. There exists a constant $c > 0$ such that,*

$$\|u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \le c(h^k + h^{r+1/2}) \quad \text{and} \quad \|u - u_h^\ell\|_{\mathrm{L}^2(\Omega,\Gamma)} \le c(h^{k+1} + h^{r+1}), \tag{17}$$

where $u_h^\ell \in \mathbb{V}_h^\ell$ denotes the lift of u_h onto Ω, given in Definition 5.

The overall error in this theorem is composed of two components: the geometrical error and the finite element error. To prove these error bounds, we proceed as follows:

1. estimate the geometric error: we bound the difference between the exact bilinear form a and the lifted bilinear form a_h^ℓ;
2. bound the H^1 error using the geometric and interpolation error estimation, proving the first inequality of (17);
3. an Aubin-Nitsche argument helps us prove the second inequality of (17).

6.1 Geometric error

First of all, we introduce $B_h^\ell \subset \Omega$ as the union of all the non-internal elements of the exact mesh $\mathcal{T}_h^{(e)}$,

$$B_h^\ell = \{\ T^{(e)} \in \mathcal{T}_h^{(e)};\ T^{(e)} \text{ has at least two vertices on } \Gamma\}.$$

Note that, by definition of B_h^ℓ, we have,

$$\frac{1}{J_h^\ell} - 1 = 0 \quad \text{and} \quad \mathcal{G}_h^{(r)} - \mathrm{Id} = 0 \quad \text{in } \Omega \backslash B_h^\ell. \tag{18}$$

The following corollary involving B_h^ℓ is a direct consequence of [18, Lemma 4.10] or [21, Theorem 1.5.1.10].

Corollary 1. *Let $v \in \mathrm{H}^1(\Omega)$ and $w \in \mathrm{H}^2(\Omega)$. Then, for a sufficiently small h, there exists $c > 0$ such that the following inequalities hold,*

$$\|v\|_{\mathrm{L}^2(B_h^\ell)} \le ch^{1/2}\|v\|_{\mathrm{H}^1(\Omega)} \qquad \text{and} \qquad \|w\|_{\mathrm{H}^1(B_h^\ell)} \le ch^{1/2}\|w\|_{\mathrm{H}^2(\Omega)}. \tag{19}$$

The difference between a and a_h, referred to as the geometric error, is evaluated in the following proposition.

Proposition 5. *Consider $v, w \in \mathbb{V}_h^\ell$. There exists $c > 0$, such that the following geometric error estimation hold,*

$$|a(v,w) - a_h^\ell(v,w)| \le ch^r \|\nabla v\|_{\mathrm{L}^2(B_h^\ell)} \|\nabla w\|_{\mathrm{L}^2(B_h^\ell)} + ch^{r+1}\|v\|_{\mathrm{H}^1(\Omega,\Gamma)}\|w\|_{\mathrm{H}^1(\Omega,\Gamma)}. \tag{20}$$

The following proof is inspired by [18, lemma 6.2]. The main difference is the use of the modified lift given in definition 5 and the corresponding transformation $G_h^{(r)}$ alongside its associated matrix $\mathcal{G}_h^{(r)}$, defined in (10), which leads to several changes in the proof.

Proof. Let $v, w \in \mathbb{V}_h^\ell$. By the definitions of the bilinear forms a and a_h^ℓ, we have,

$$|a(v,w) - a_h^\ell(v,w)| \le a_1(v,w) + \kappa a_2(v,w) + \beta a_3(v,w) + \alpha a_4(v,w),$$

where the terms a_i, defined on $\mathbb{V}_h^\ell \times \mathbb{V}_h^\ell$, are respectively given by,

$$a_1(v,w) := \left| \int_\Omega \nabla w \cdot \nabla v - \mathcal{G}_h^{(r)} \nabla w \cdot \mathcal{G}_h^{(r)} \nabla v \frac{1}{J_h^\ell} \,\mathrm{d}x \right|, \quad a_2(v,w) := \left| \int_\Omega wv \, (1 - \frac{1}{J_h^\ell}) \,\mathrm{d}x \right|,$$
$$a_3(v,w) := \left| \int_\Gamma (A_h^\ell - \mathrm{Id}) \, \nabla_\Gamma w \cdot \nabla_\Gamma v \,\mathrm{d}\sigma \right|, \qquad a_4(v,w) := \left| \int_\Gamma wv \, (1 - \frac{1}{J_b^\ell}) \,\mathrm{d}\sigma \right|.$$

The next step is to bound each a_i, for $i = 1,2,3,4$, while using (13) and (12).

First of all, notice that $a_1(v,w) \le Q_1 + Q_2 + Q_3$, where,

$$Q_1 := \left| \int_\Omega (\mathcal{G}_h^{(r)} - \mathrm{Id}) \, \nabla w \cdot \mathcal{G}_h^{(r)} \nabla v \frac{1}{J_h^\ell} \,\mathrm{d}x \right|,$$
$$Q_2 := \left| \int_\Omega \nabla w \cdot (\mathcal{G}_h^{(r)} - \mathrm{Id}) \nabla v \frac{1}{J_h^\ell} \,\mathrm{d}x \right|,$$
$$Q_3 := \left| \int_\Omega \nabla w \cdot \nabla v (\frac{1}{J_h^\ell} - 1) \,\mathrm{d}x \right|.$$

We use (18) and (13) to estimate each Q_j as follows,

$$Q_1 = \left| \int_{B_h^\ell} (\mathcal{G}_h^{(r)} - \mathrm{Id}) \, \nabla w \cdot \mathcal{G}_h^{(r)} \nabla v \frac{1}{J_h^\ell} \,\mathrm{d}x \right| \le ch^r \|\nabla w\|_{\mathrm{L}^2(B_h^\ell)} \|\nabla v\|_{\mathrm{L}^2(B_h^\ell)},$$
$$Q_2 = \left| \int_{B_h^\ell} \nabla w \cdot (\mathcal{G}_h^{(r)} - \mathrm{Id}) \nabla v \frac{1}{J_h^\ell} \,\mathrm{d}x \right| \le ch^r \|\nabla w\|_{\mathrm{L}^2(B_h^\ell)} \|\nabla v\|_{\mathrm{L}^2(B_h^\ell)},$$
$$Q_3 = \left| \int_{B_h^\ell} \nabla w \cdot \nabla v (\frac{1}{J_h^\ell} - 1) \,\mathrm{d}x \right| \le ch^r \|\nabla w\|_{\mathrm{L}^2(B_h^\ell)} \|\nabla v\|_{\mathrm{L}^2(B_h^\ell)}.$$

Summing up the latter terms, we get, $a_1(v,w) \le ch^r \|\nabla w\|_{\mathrm{L}^2(B_h^\ell)} \|\nabla v\|_{\mathrm{L}^2(B_h^\ell)}$.

Similarly, to bound a_2, we proceed by using (18) and (13) as follows,

$$a_2(v,w) = \left| \int_{B_h^\ell} wv \, (1 - \frac{1}{J_h^\ell}) \,\mathrm{d}x \right| \le ch^r \|w\|_{\mathrm{L}^2(B_h^\ell)} \|v\|_{\mathrm{L}^2(B_h^\ell)}.$$

Since $v, w \in \mathbb{V}_h^\ell \subset \mathrm{H}^1(\Omega,\Gamma)$, we use (19) to get,

$$a_2(v,w) \le ch^{r+1} \|w\|_{\mathrm{H}^1(\Omega)} \|v\|_{\mathrm{H}^1(\Omega)}.$$

Before estimating a_3, we need to notice that, by definition of the tangential gradient over Γ, $P\nabla_\Gamma = \nabla_\Gamma$ where $P = \mathrm{Id} - \mathbf{n}\otimes\mathbf{n}$ is the orthogonal projection over the tangential spaces of Γ. With the estimation (12), we get,

$$\begin{aligned} a_3(v,w) &= \left| \int_\Gamma (A_h^\ell - P)\ \nabla_\Gamma w \cdot \nabla_\Gamma v \,\mathrm{d}\sigma \right| \\ &\leq ||A_h^\ell - P||_{\mathrm{L}^\infty(\Gamma)} \|w\|_{\mathrm{H}^1(\Gamma)} \|v\|_{\mathrm{H}^1(\Gamma)} \leq ch^{r+1} \|w\|_{\mathrm{H}^1(\Gamma)} \|v\|_{\mathrm{H}^1(\Gamma)}. \end{aligned}$$

Finally, using (12), we estimate a_4 as follows,

$$a_4(v,w) = \left| \int_\Gamma wv\ (1 - \frac{1}{J_b^\ell})\,\mathrm{d}\sigma \right| \leq ch^{r+1} \|w\|_{\mathrm{L}^2(\Gamma)} \|v\|_{\mathrm{L}^2(\Gamma)}.$$

The inequality (20) is easy to obtain when summing up a_i, for all $i = 1,2,3,4$. □

Remark 7. *Let us point out that, with u (resp. u_h) the solution of the problem* (2) *(resp.* (15)*), we have,*

$$\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \leq c\|u\|_{\mathrm{H}^1(\Omega,\Gamma)}, \tag{21}$$

where $c > 0$ is independent with respect to h. In fact, a relatively easy way to prove it is by employing the geometrical error estimation (20)*, as follows,*

$$c_c \|u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} \leq a(u_h^\ell, u_h^\ell) \leq a(u_h^\ell, u_h^\ell) - a(u, u_h^\ell) + a(u, u_h^\ell),$$

where c_c is the coercivity constant. Using (16)*, we have,*

$$c_c \|u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} \leq a(u_h^\ell, u_h^\ell) - a_h^\ell(u_h^\ell, u_h^\ell) + a(u, u_h^\ell) = (a - a_h^\ell)(u_h^\ell, u_h^\ell) + a(u, u_h^\ell).$$

Thus applying the estimation (20) *along with the continuity of a, we get,*

$$\begin{aligned} \|u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} &\leq ch^r \|\nabla u_h^\ell\|^2_{\mathrm{L}^2(B_h^\ell)} + ch^{r+1} \|u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} + c\|u\|_{\mathrm{H}^1(\Omega,\Gamma)} \|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \\ &\leq ch^r \|u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} + c\|u\|_{\mathrm{H}^1(\Omega,\Gamma)} \|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}. \end{aligned}$$

Thus, we have,

$$(1 - ch^r) \|u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} \leq c\|u\|_{\mathrm{H}^1(\Omega,\Gamma)} \|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}.$$

For a sufficiently small h, we have $1 - ch^r > 0$, which concludes the proof.

6.2 Proof of the H^1 error bound in Theorem 2

Let $u \in \mathrm{H}^{k+1}(\Omega,\Gamma)$ and $u_h \in \mathbb{V}_h$ be the respective solutions of (2) and (15).

To begin with, we use the coercivity of the bilinear form a to obtain, denoting c_c as the coercivity constant,

$$\begin{aligned} c_c \|\mathcal{I}^\ell u - u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} &\leq a(\mathcal{I}^\ell u - u_h^\ell, \mathcal{I}^\ell u - u_h^\ell) = a(\mathcal{I}^\ell u, \mathcal{I}^\ell u - u_h^\ell) - a(u_h^\ell, \mathcal{I}^\ell u - u_h^\ell) \\ &= a_h^\ell(u_h^\ell, \mathcal{I}^\ell u - u_h^\ell) - a(u_h^\ell, \mathcal{I}^\ell u - u_h^\ell) + a(\mathcal{I}^\ell u, \mathcal{I}^\ell u - u_h^\ell) - a_h^\ell(u_h^\ell, \mathcal{I}^\ell u - u_h^\ell), \end{aligned}$$

where. in the latter equation, we added and subtracted $a_h^\ell(u_h^\ell, \mathcal{I}^\ell u - u_h^\ell)$. Thus,

$$c_c \|\mathcal{I}^\ell u - u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} \leq \big(a_h^\ell - a\big)(u_h^\ell, \mathcal{I}^\ell u - u_h^\ell) + a(\mathcal{I}^\ell u, \mathcal{I}^\ell u - u_h^\ell) - a_h^\ell(u_h^\ell, \mathcal{I}^\ell u - u_h^\ell).$$

Applying (16) with $v = \mathcal{I}^\ell u - u_h^\ell \in \mathbb{V}_h^\ell$, we have,

$$c_c \|\mathcal{I}^\ell u - u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} \leq |(a_h^\ell - a)(u_h^\ell, \mathcal{I}^\ell u - u_h^\ell)| + |a(\mathcal{I}^\ell u - u, \mathcal{I}^\ell u - u_h^\ell)|.$$

Taking advantage of the continuity of a and the estimation (20), we obtain,

$$\begin{aligned}
&c_c\|\mathcal{I}^\ell u - u_h^\ell\|^2_{\mathrm{H}^1(\Omega,\Gamma)} \\
&\quad\le\ c\big(h^r\|\nabla u_h^\ell\|_{\mathrm{L}^2(B_h^\ell)}\|\nabla(\mathcal{I}^\ell u - u_h^\ell)\|_{\mathrm{L}^2(B_h^\ell)} + h^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}\|\mathcal{I}^\ell u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}\big) \\
&\qquad\qquad + c_{cont}\|\mathcal{I}^\ell u - u\|_{\mathrm{H}^1(\Omega,\Gamma)}\|\mathcal{I}^\ell u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \\
&\quad\le\ c\big(h^r\|\nabla u_h^\ell\|_{\mathrm{L}^2(B_h^\ell)} + h^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \\
&\qquad\qquad + c_{cont}\|\mathcal{I}^\ell u - u\|_{\mathrm{H}^1(\Omega,\Gamma)}\big)\|\mathcal{I}^\ell u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}.
\end{aligned}$$

Then, dividing by $\|\mathcal{I}^\ell u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}$, we have,

$$\|\mathcal{I}^\ell u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \le c\left(h^r\|\nabla u_h^\ell\|_{\mathrm{L}^2(B_h^\ell)} + h^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} + \|\mathcal{I}^\ell u - u\|_{\mathrm{H}^1(\Omega,\Gamma)}\right).$$

To conclude, we use the latter inequality in the following estimation as follows,

$$\begin{aligned}
\|u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} &\le \|u - \mathcal{I}^\ell u\|_{\mathrm{H}^1(\Omega,\Gamma)} + \|\mathcal{I}^\ell u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \\
&\le c\left(h^r\|\nabla u_h^\ell\|_{\mathrm{L}^2(B_h^\ell)} + h^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} + \|\mathcal{I}^\ell u - u\|_{\mathrm{H}^1(\Omega,\Gamma)}\right)
\end{aligned}$$

Using the proposition 4 and the inequalities (19), we have,

$$\begin{aligned}
&\|u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \\
&\le ch^r(\|\nabla(u_h^\ell - u)\|_{\mathrm{L}^2(B_h^\ell)} + \|\nabla u\|_{\mathrm{L}^2(B_h^\ell)}) + ch^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} + ch^k\|u\|_{\mathrm{H}^{k+1}(\Omega,\Gamma)} \\
&\le ch^r(\|u_h^\ell - u\|_{\mathrm{H}^1(\Omega,\Gamma)} + h^{1/2}\|u\|_{\mathrm{H}^2(\Omega)}) + ch^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} + ch^k\|u\|_{\mathrm{H}^{k+1}(\Omega,\Gamma)}.
\end{aligned}$$

Thus we have,

$$(1 - ch^r)\|u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \le c\left(h^{r+1/2}\|u\|_{\mathrm{H}^2(\Omega)} + h^k\|u\|_{\mathrm{H}^{k+1}(\Omega,\Gamma)} + h^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}\right).$$

For a sufficiently small h, we arrive at,

$$\|u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \le c\left(h^{r+1/2}\|u\|_{\mathrm{H}^2(\Omega,\Gamma)} + h^k\|u\|_{\mathrm{H}^{k+1}(\Omega,\Gamma)} + h^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}\right).$$

This provides the desired result using (21).

6.3 Proof of the L^2 error bound in Theorem 2

Recall that $u \in \mathrm{H}^1(\Omega,\Gamma)$ is the solution of the variational problem (2), $u_h \in \mathbb{V}_h$ is the solution of the discrete problem (15). To estimate the L^2 norm of the error, we define the functional F_h by,

$$\begin{array}{rccl} F_h: & \mathrm{H}^1(\Omega,\Gamma) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & F_h(v) = a(u - u_h^\ell, v). \end{array}$$

We bound $|F_h(v)|$ for any $v \in \mathrm{H}^2(\Omega,\Gamma)$ in the lemma 1. Afterwards an Aubin-Nitsche argument is applied to bound the L^2 norm of the error.

Lemma 1. *For all $v \in \mathrm{H}^2(\Omega,\Gamma)$, there exists $c > 0$ such that the following inequality holds,*

$$|F_h(v)| \le c(h^{k+1} + h^{r+1})\|v\|_{\mathrm{H}^2(\Omega,\Gamma)}. \tag{22}$$

Remark 8. *To prove lemma 1, some key points for a function $v \in \mathrm{H}^2(\Omega,\Gamma)$ are presented. Firstly, inequality* (19) *implies that,*

$$\forall\, v \in \mathrm{H}^2(\Omega,\Gamma), \quad \|\nabla v\|_{\mathrm{L}^2(B_h^\ell)} \le ch^{1/2}\|v\|_{\mathrm{H}^2(\Omega)}. \tag{23}$$

Secondly, then the interpolation inequality in proposition 4 gives,

$$\forall\, v \in \mathrm{H}^2(\Omega,\Gamma), \quad \|\mathcal{I}^\ell v - v\|_{\mathrm{H}^1(\Omega,\Gamma)} \le ch\|v\|_{\mathrm{H}^2(\Omega,\Gamma)}. \tag{24}$$

Applying 16 for $\mathcal{I}^\ell v \in \mathbb{V}_h^\ell$, we have,

$$\forall\, v \in \mathrm{H}^2(\Omega,\Gamma), \quad a(u, \mathcal{I}^\ell v) = l(\mathcal{I}^\ell v) = a_h^\ell(u_h^\ell, \mathcal{I}^\ell v). \tag{25}$$

Proof of lemma 1. Consider $v \in \mathrm{H}^2(\Omega,\Gamma)$. We may decompose $|F_h(v)|$ in two terms as follows,

$$|F_h(v)| = |a(u - u_h^\ell, v)| \le |a(u - u_h^\ell, v - \mathcal{I}^\ell v)| + |a(u - u_h^\ell, \mathcal{I}^\ell v)| =: F_1 + F_2.$$

Firstly, to bound F_1, we take advantage of the continuity of the bilinear form a and apply the H^1 error estimation (17), alongside the inequality (24) as follows,

$$\begin{aligned} F_1 &\le c_{cont}\, \|u - u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \|v - \mathcal{I}^\ell v\|_{\mathrm{H}^1(\Omega,\Gamma)} \le c(h^k + h^{r+1/2})\, h\|v\|_{\mathrm{H}^2(\Omega,\Gamma)} \\ &\le c(h^{k+1} + h^{r+3/2})\, \|v\|_{\mathrm{H}^2(\Omega,\Gamma)}. \end{aligned}$$

Secondly, to estimate F_2, we resort to equations (25) and (20) as follows,

$$\begin{aligned} F_2 = |a(u, \mathcal{I}^\ell v) - a(u_h^\ell, \mathcal{I}^\ell v)| &= |a_h^\ell(u_h^\ell, \mathcal{I}^\ell v) - a(u_h^\ell, \mathcal{I}^\ell v)| = |(a_h^\ell - a)(u_h^\ell, \mathcal{I}^\ell v)| \\ &\le ch^r \|\nabla u_h^\ell\|_{\mathrm{L}^2(B_h^\ell)} \|\nabla(\mathcal{I}^\ell v)\|_{\mathrm{L}^2(B_h^\ell)} + ch^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)} \|\mathcal{I}^\ell v\|_{\mathrm{H}^1(\Omega,\Gamma)}. \end{aligned}$$

Next, we will treat the first term in the latter inequality separately. We have,

$$\begin{aligned} F_3 &:= h^r \|\nabla u_h^\ell\|_{\mathrm{L}^2(B_h^\ell)} \|\nabla(\mathcal{I}^\ell v)\|_{\mathrm{L}^2(B_h^\ell)} \\ &\le h^r \Big(\|\nabla(u_h^\ell - u)\|_{\mathrm{L}^2(B_h^\ell)} + \|\nabla u\|_{\mathrm{L}^2(B_h^\ell)}\Big)\Big(\|\nabla(\mathcal{I}^\ell v - v)\|_{\mathrm{L}^2(B_h^\ell)} + \|\nabla v\|_{\mathrm{L}^2(B_h^\ell)}\Big) \\ &\le h^r \Big(\|u_h^\ell - u\|_{\mathrm{H}^1(\Omega,\Gamma)} + \|\nabla u\|_{\mathrm{L}^2(B_h^\ell)}\Big)\Big(\|\mathcal{I}^\ell v - v\|_{\mathrm{H}^1(\Omega,\Gamma)} + \|\nabla v\|_{\mathrm{L}^2(B_h^\ell)}\Big). \end{aligned}$$

We now apply the H^1 error estimation (17), the inequality (23) and the interpolation inequality (24), as follows,

$$\begin{aligned} F_3 &\le c\, h^r \Big(h^k + h^{r+1/2} + h^{1/2}\|u\|_{\mathrm{H}^2(\Omega,\Gamma)}\Big)\Big(h\|v\|_{\mathrm{H}^2(\Omega,\Gamma)} + h^{1/2}\|v\|_{\mathrm{H}^2(\Omega,\Gamma)}\Big) \\ &\le c\, h^r\, h^{1/2}\Big(h^{k-1/2} + h^r + \|u\|_{\mathrm{H}^2(\Omega,\Gamma)}\Big)\Big(h^{1/2} + 1\Big) h^{1/2}\|v\|_{\mathrm{H}^2(\Omega,\Gamma)} \\ &\le c\, h^{r+1}\Big(h^{k-1/2} + h^r + \|u\|_{\mathrm{H}^2(\Omega,\Gamma)}\Big)\Big(h^{1/2} + 1\Big)\|v\|_{\mathrm{H}^2(\Omega,\Gamma)}. \end{aligned}$$

Noticing that $k - 1/2 > 0$ (since $k \ge 1$) and that $\Big(h^{k-1/2} + h^r + \|u\|_{\mathrm{H}^2(\Omega,\Gamma)}\Big)\Big(h^{1/2} + 1\Big)$ is bounded by a constant independent of h, we obtain $F_3 \le c\, h^{r+1}\|v\|_{\mathrm{H}^2(\Omega,\Gamma)}$. Using the previous expression of F_2,

$$F_2 \le ch^{r+1}\|v\|_{\mathrm{H}^2(\Omega,\Gamma)} + ch^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}\|\mathcal{I}^\ell v\|_{\mathrm{H}^1(\Omega,\Gamma)}.$$

Moreover, noticing that $\|\mathcal{I}^\ell v\|_{\mathrm{H}^1(\Omega,\Gamma)} \le c\|v\|_{\mathrm{H}^2(\Omega,\Gamma)}$,

$$F_2 \le ch^{r+1}\|v\|_{\mathrm{H}^2(\Omega,\Gamma)} + ch^{r+1}\|u_h^\ell\|_{\mathrm{H}^1(\Omega,\Gamma)}\|v\|_{\mathrm{H}^2(\Omega,\Gamma)} \le ch^{r+1}\|v\|_{\mathrm{H}^2(\Omega,\Gamma)},$$

using (21). We conclude the proof by summing the estimates of F_1 and F_2. □

Proof of the L^2 *estimate* (17). Defining $e := u - u_h^\ell$, the aim is to estimate the following L^2 error norm: $\|e\|^2_{L^2(\Omega,\Gamma)} = \|u - u_h^\ell\|^2_{L^2(\Omega)} + \|u - u_h^\ell\|^2_{L^2(\Gamma)}$. Let $v \in L^2(\Omega, \Gamma)$. We define the following problem: find $z_v \in H^1(\Omega, \Gamma)$ such that,

$$a(w, z_v) = \langle w, v \rangle_{L^2(\Omega,\Gamma)}, \quad \forall\, w \in H^1(\Omega, \Gamma), \tag{26}$$

Applying Theorem 1 for $f = v$ and $g = v_{|\Gamma}$, there exists a unique solution $z_v \in H^1(\Omega, \Gamma)$ to (26), which satisfies the following inequality,

$$\|z_v\|_{H^2(\Omega,\Gamma)} \leq c\|v\|_{L^2(\Omega,\Gamma)}.$$

Taking $v = e \in L^2(\Omega, \Gamma)$ and $w = e \in H^1(\Omega, \Gamma)$ in (26), we obtain $F_h(z_e) = a(e, z_e) = \|e\|^2_{L^2(\Omega,\Gamma)}$. In this case, Theorem 1 implies,

$$\|z_e\|_{H^2(\Omega,\Gamma)} \leq c\|e\|_{L^2(\Omega,\Gamma)}. \tag{27}$$

Applying Inequality (22) for $z_e \in H^2(\Omega, \Gamma)$ and afterwards Inequality (27), we have,

$$\|e\|^2_{L^2(\Omega,\Gamma)} = |F_h(z_e)| \leq c(h^{k+1} + h^{r+1})\|z_e\|_{H^2(\Omega,\Gamma)} \leq c(h^{k+1} + h^{r+1})\|e\|_{L^2(\Omega,\Gamma)},$$

which concludes the proof. □

7 Numerical experiments

In this section are presented numerical results aimed to illustrate the theoretical convergence results in theorem 2. Supplementary numerical results will be provided in order to highlight the properties of the volume lift introduced in definition 5 relatively to the lift transformation $G_h^{(r)} : \Omega_h^{(r)} \to \Omega$ given in (4).

The Ventcel problem (1) is considered with $\alpha = \beta = \kappa = 1$ on the unit disk Ω,

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ -\Delta_\Gamma u + \partial_n u + u = g & \text{on } \Gamma, \end{cases}$$

with the source terms $f(x, y) = -y e^x$ and $g(x, y) = y e^x(3 + 4x - y^2)$ corresponding to the exact solution $u = -f$. The discrete problem (15) is implemented and solved using the finite element library Cumin [26]. Curved meshes of Ω of geometrical order $1 \leq r \leq 3$ have been generated using the software Gmsh[1]. All numerical results presented in this section can be fully reproduced using dedicated source codes available on Cumin Gitlab[2].

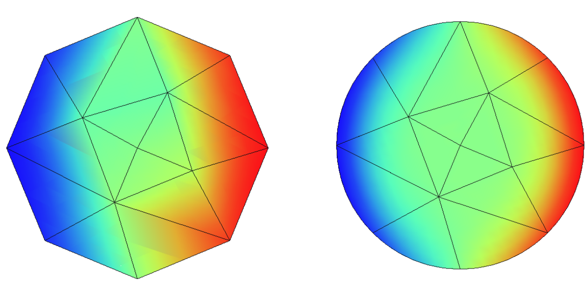

Figure 3: Numerical solution of the Ventcel problem on affine and quadratic meshes.

[1]Gmsh: a three-dimensional finite element mesh generator, `https://gmsh.info/`

[2]Cumin GitLab deposit, `https://gmsh.info/`

The numerical solutions u_h are computed for $\mathbb{P}^k$ finite elements, with $k = 1, \dots, 4$, on series of successively refined meshes of order $r = 1, \dots, 3$, as depicted on figure 3 for coarse meshes (affine and quadratic). For each mesh order r and each finite element degree k, the following numerical errors are computed:

$$\|u - u_h^\ell\|_{\mathrm{L}^2(\Omega)}, \quad \|\nabla u - \nabla u_h^\ell\|_{\mathrm{L}^2(\Omega)}, \quad \|u - u_h^\ell\|_{\mathrm{L}^2(\Gamma)} \quad \text{and} \quad \|\nabla_\Gamma u - \nabla_\Gamma u_h^\ell\|_{\mathrm{L}^2(\Gamma)},$$

with u_h^ℓ given in definition 5. The convergence orders of these errors, relatively to the mesh size, are reported in tables 1 and 2.

	$\|u - u_h^\ell\|_{\mathrm{L}^2(\Omega)}$				$\|\nabla u - \nabla u_h^\ell\|_{\mathrm{L}^2(\Omega)}$			
	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$
Affine mesh (r=1)	1.98	1.99	1.97	1.97	1.00	1.50	1.49	1.49
Quadratic mesh (r=2)	2.01	3.14	3.94	3.97	1.00	2.12	3.03	3.48
Cubic mesh (r=3)	2.04	2.45	3.44	4.04	1.02	1.47	2.42	3.46

Table 1: Convergence orders, interior norms.

The convergence orders presented in table 1, relatively to L^2 norms on Ω, deserve comments. In the affine case $r = 1$, the figures are in perfect agreement with estimates (17): the L^2 error norm is in $O(h^{k+1} + h^2)$ and the L^2 norm of the gradient of the error is in $O(h^k + h^{1.5})$. For quadratic meshes, a super convergence is observed, the case $r = 2$ behaves as if $r = 3$: the L^2 error norm is in $O(h^{k+1} + h^4)$ and the L^2 norm of the gradient of the error is in $O(h^k + h^{3.5})$. This super convergence, though not understood, has been documented and further investigated in [8]. For the cubic case eventually, a default of order -1/2 is observed on the convergence orders (excepted for the $\mathbb{P}^1$ case): the L^2 error norm is in $O(h^{k+1/2} + h^4)$ and the L^2 norm of the gradient of the error is in $O(h^{k-1/2} + h^{3.5})$. This default might not be in relation with the finite element approximation since it is not observed on the quadratic case but might neither be related with the cubic meshes since this default vanishes when considering L^2 boundary errors, see table 2. Further experiments showed us that this default is not caused by the specific Ventcel boundary condition, it similarly occurs when considering a Poisson problem with Neumann boundary condition on the disk: so far we have no clues on its explanation.

	$\|u - u_h^\ell\|_{\mathrm{L}^2(\Gamma)}$				$\|\nabla_\Gamma u - \nabla_\Gamma u_h^\ell\|_{\mathrm{L}^2(\Gamma)}$			
	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$
Affine mesh (r=1)	2.00	2.03	2.01	2.01	1.00	2.00	1.98	1.98
Quadratic mesh (r=2)	2.00	3.00	4.00	4.02	1.00	2.00	3.00	4.02
Cubic mesh (r=3)	2.00	3.00	4.00	4.21	1.00	2.00	3.00	3.98

Table 2: Convergence orders, boundary norms.

Let us now discuss table 2, which reports convergence orders relatively to L^2 boundary norms. The first interesting point is that the L^2 convergence towards the gradient of u is faster than expressed in (17): $O(h^k + h^{r+1})$ instead of $O(h^k + h^{r+1/2})$. Meanwhile the L^2 convergence towards u behaves as expected. The super convergence in the quadratic case is similarly observed, which meshes behave as if $r = 3$.

Lift transformation regularity As discussed in remark 3, the lift transformation $G_h^{(r)} : \Omega_h^{(r)} \to \Omega$ in (4) has a regularity controlled by the exponent s in the term $(\lambda^\star)^s$, which exponent is set to $r + 2$ to ensure a piecewise C^{r+1} regularity. The convergence properties should however remain the same when setting $s = 2$. This has been numerically observed: the tables 1 and 2 remain identical when setting $s = 2$. More surprisingly, when setting $s = 1$, tables 1 and 2 also remain unchanged, though in this case estimate (5) does no longer hold. Precisely in this case, it can be easily seen that $\mathrm{D}G_h^{(r)}$ has singularities in the non-internal elements. It seems that this singularities "*are not seen*", likely because the quadrature method nodes (used to approximate integrals) lie away of these singularities.

	$\|u - u_h^\ell\|_{\mathrm{L}^2(\Omega)}$				$\|\nabla u - \nabla u_h^\ell\|_{\mathrm{L}^2(\Omega)}$			
	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$
Quadratic mesh (r=2)	2.01	2.51	2.49	2.49	1.00	1.52	1.49	1.49
Cubic mesh (r=3)	2.04	2.50	2.48	2.49	1.03	1.51	1.49	1.49

	$\|u - u_h^\ell\|_{\mathrm{L}^2(\Gamma)}$				$\|\nabla_\Gamma u - \nabla_\Gamma u_h^\ell\|_{\mathrm{L}^2(\Gamma)}$			
	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$	$\mathbb{P}^1$	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}^4$
Quadratic mesh (r=2)	2.00	3.00	2.99	2.99	1.00	2.00	3.00	2.98
Cubic mesh (r=3)	2.00	3.00	2.99	2.98	1.00	2.00	3.00	2.98

Table 3: Convergence orders for the lift in [18].

Former lift definition As developed in remark 4, another lift transformation $G_h^{(r)} : \Omega_h^{(r)} \to \Omega$ had formerly been introduced in [18], with different properties on the boundary. We reported the convergence orders observed with this lift in table 3.
The first observation is that $\|u - u_h^\ell\|_{\mathrm{L}^2(\Omega)}$ is at most in $O(h^{2.5})$ whereas $\|\nabla u - \nabla u_h^\ell\|_{\mathrm{L}^2(\Omega)}$ is at most in $O(h^{1.5})$, resulting in a clear decrease of the convergence rate as compared to tables 1 and 2. Similarly, $\|u - u_h^\ell\|_{\mathrm{L}^2(\Gamma)}$ and $\|\nabla u - \nabla u_h^\ell\|_{\mathrm{L}^2(\Gamma)}$ are at most in $O(h^3)$ whereas they could reach $O(h^4)$ in tables 1 and 2.

Notice that the lift transformation intervenes at two different stages: for the right hand side definition in (15) and for the error computation itself. We experienced the following. We set the lift for the right hand side computation to the one in [18] whereas the lift for the error computation is the one in definition 5 (so that the numerical solution u_h is the same as in table 3, only its post treatment in terms of errors is different). Then we observed that the results are partially improved: for the $\mathbb{P}^4$ case on cubic meshes, $\|u - u_h^\ell\|_{\mathrm{L}^2(\Omega)} = O(h^{3.0})$ and $\|\nabla u - \nabla u_h^\ell\|_{\mathrm{L}^2(\Omega)} = O(h^{2.5})$, which remains lower than the convergence orders in table 1.

Still considering the lift definition in [18], we also experienced that the exponent s in the term $(\lambda^\star)^s$ in the lift definition (see remark 3) has an influence on the convergence rates. Surprisingly, the best convergence rates are obtained when setting $s = 1$: this case corresponds to the minimal regularity on the lift transformation $G_h^{(r)}$, the differential of which (as previously discussed) has singularities on the non-internal mesh elements. In that case however, the convergence rares goes up to $O(h^{3.5})$ and $O(h^{2.5})$ on quadratic and cubic meshes for $\|u - u_h^\ell\|_{\mathrm{L}^2(\Omega)}$ and $\|\nabla u - \nabla u_h^\ell\|_{\mathrm{L}^2(\Omega)}$ respectively. Meanwhile, it has been noticed that setting $s = 1$ somehow damages the quality of the numerical solution on the domain boundary: these last results are surprising and with no clear explanation. Eventually, when setting $s \geq 2$, the convergence rates are lower and identical to those in table 3.

A Proof of Proposition 3

Following the notations given in definition 2, we present the proof of Proposition 3 which requires a series of preliminary results given in Propositions 6, 7 and 8. The proofs of these propositions are inspired by the proofs of [2, lemma 6.2], [18, lemma 4.3] and [18, proposition 4.4] respectively.

Proposition 6. *The map $y : \hat{x} \in \hat{T}\backslash\hat{\sigma} \mapsto y := F_T^{(r)}(\hat{y}) \in \Gamma_h^{(r)}$ is a smooth function and for all $m \geq 1$, there exists a constant $c > 0$ independent of h such that,*

$$\|\mathrm{D}^m y\|_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} \leq \frac{ch}{(\lambda^*)^m}. \tag{28}$$

Remark 9. *The proof of this proposition and of the next one rely on the formula of Faà di Bruno (see [2, equation 2.9]). This formula states that for two functions f and g, which are of class $\mathcal{C}^m$, such that $f \circ g$ is well defined, then,*

$$\mathrm{D}^m(f \circ g) = \sum_{p=1}^{m} \Big(\mathrm{D}^p(f) \sum_{i \in E(m,p)} c_i \prod_{q=1}^{m} \mathrm{D}^q g^{i_q} \Big), \tag{29}$$

where $E(m,p) := \{i \in \mathbb{N}^m; \sum_{q=1}^m i_q = p \text{ and } \sum_{q=1}^m q i_q = m\}$ and c_i are positives constants, for all $i \in E(m,p)$.

Proof of Proposition 6. We detail the proof in the 2 dimensional case, the 3D case can be proved in a similar way.

Consider, the reference triangle $\hat{T}$ with the usual orientation. Its vertices are denoted $(\hat{v}_i)_{i=1}^3$ and the associated barycentric coordinates respectively are: $\lambda_1 = 1 - x_1 - x_2$, $\lambda_2 = x_2$ and $\lambda_3 = x_1$. Consider a non-internal mesh element $T^{(r)}$ such that, without loss of generality, $v_1 \notin \Gamma$. In such a case, depicted in figure 4, $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon_3 = 1$, since $v_2, v_3 \in \Gamma \cap T^{(r)}$. This implies that $\lambda^* = \lambda_2 + \lambda_3 = x_2 + x_1$ and,

$$\hat{y} = \frac{1}{\lambda^*}(\lambda_2 \hat{v}_2 + \lambda_3 \hat{v}_3) = \frac{1}{x_2 + x_1}(x_2 \hat{v}_2 + x_1 \hat{v}_3). \tag{30}$$

In this case, $\hat{\sigma} = \{\hat{v}_1\}$ and $\hat{y}$ is defined on $\hat{T} \setminus \{\hat{v}_1\}$.

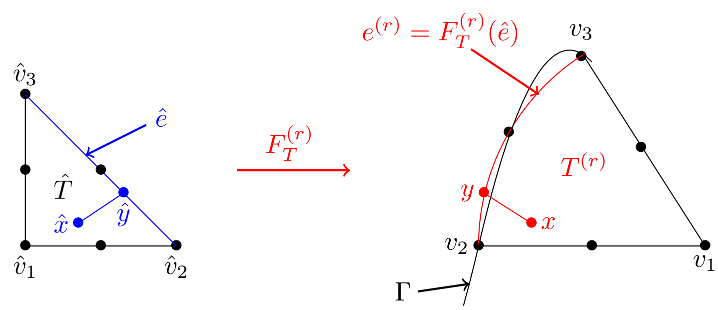

Figure 4: Displaying $F_T^{(r)} : \hat{T} \to T^{(r)}$ in a 2D quadratic case (r=2).

By differentiating the expression (30) of $\hat{y}$ and using an induction argument, it can be proven that there exists a constant $c > 0$, independent of h, such that,

$$\|\mathrm{D}^m \hat{y}\|_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} \leq \frac{c}{(\lambda^*)^m}, \qquad \text{for all } m \geq 1. \tag{31}$$

Since $F_T^{(r)}$ is the $\mathbb{P}^r$-Lagrangian interpolant of $F_T^{(e)}$ on $\hat{T}$, then $y = F_T^{(r)} \circ \hat{y}$ is a smooth function on $\hat{T}\backslash\hat{\sigma}$. We now apply the inequality (29) for $y = F_T^{(r)} \circ \hat{y}$ to estimate its derivative's norm as follows, for all $m \geq 1$,

$$\|\mathrm{D}^m(y)\|_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} \leq \sum_{p=1}^{m} \Big(\|\mathrm{D}^p(F_T^{(r)})\|_{\mathrm{L}^\infty(\hat{e})} \sum_{i \in E(m,p)} c_i \prod_{q=1}^{m} \|\mathrm{D}^q \hat{y}\|^{i_q}_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} \Big),$$

where $\hat{e} := (F_T^{(r)})^{(-1)}(e^{(r)})$ and $e^{(r)} := \partial T^{(r)} \cap \Gamma_h^{(r)}$ are displayed in Figure 4. Afterwards, we decompose the sum into two parts, one part taking $p = 1$ and the second one for $p \geq 2$, and apply inequality (31),

$$\begin{aligned}
&\|\mathrm{D}^m(y)\|_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} \\
&\quad \leq \|\mathrm{D}(F_T^{(r)})\|_{\mathrm{L}^\infty(\hat{e})} \sum_{i \in E(m,1)} \prod_{q=1}^{m} (\frac{c}{(\lambda^*)^q})^{i_q} + \sum_{p=2}^{m} \Big(\|\mathrm{D}^p(F_T^{(r)})\|_{\mathrm{L}^\infty(\hat{e})} \sum_{i \in E(m,p)} \prod_{q=1}^{m} (\frac{c}{(\lambda^*)^q})^{i_q} \Big) \\
&\quad \leq ch\lambda^{*(-\sum_{q=1}^m q i_q)} + c\sum_{p=2}^{m} h^r \lambda^{*(-\sum_{q=1}^m q i_q)} \leq ch(\lambda^*)^{-m},
\end{aligned}$$

using that $\|\mathrm{D}(F_T^{(r)})\|_{\mathrm{L}^\infty(\hat{e})} \leq ch$ and $\|\mathrm{D}^p(F_T^{(r)})\|_{\mathrm{L}^\infty(\hat{e})} \leq ch^r$, for $2 \leq p \leq r+1$ (see [10, page 239]), where the constant $c > 0$ is independent of h. This concludes the proof. □

Proposition 7. *Assume that* Γ *is* $\mathcal{C}^{r+2}$ *regular. Then the mapping* $b \circ y : \hat{x} \in \hat{T}\backslash\hat{\sigma} \mapsto b(y(\hat{x})) \in \Gamma$ *is of class* $\mathcal{C}^{r+1}$*. Additionally, for any* $1 \leq m \leq r+1$*, there exists a constant* $c > 0$ *independent of* h *such that,*

$$\|\mathrm{D}^m(b(y) - y)\|_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} \leq \frac{ch^{r+1}}{(\lambda^*)^m}. \tag{32}$$

Proof. Since Γ is $\mathcal{C}^{r+2}$ regular, the orthogonal projection b is a $\mathcal{C}^{r+1}$ function on a tubular neighborhood of Γ (see [16, Lemma 4.1] or [4]). Consequently, following Proposition 6, $b(y) - y$ is of class $\mathcal{C}^{r+1}$ on $\hat{T}\backslash\hat{\sigma}$.

Secondly, consider $1 \leq m \leq r+1$. Applying the Faà di Bruno formula (29) for the function $b(y) - y = (b - id) \circ y$, we have,

$$\|\mathrm{D}^m(b(y) - y)\|_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} \leq \sum_{p=1}^{m} \Big(\|\mathrm{D}^p(b - id)\|_{\mathrm{L}^\infty(e^{(r)})} \sum_{i \in E(m,p)} c_i \prod_{q=1}^{m} \|\mathrm{D}^q y\|^{i_q}_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} \Big), \tag{33}$$

where $e^{(r)} = \partial T^{(r)} \cap \Gamma_h^{(r)}$ is displayed in Figure 4. Notice that $b(v) = v$ for any $\mathbb{P}^r$-Lagrangian interpolation nodes $v \in \Gamma \cap e^{(r)}$. Then $id_{|_{e^{(r)}}}$ is the $\mathbb{P}^r$-Lagrangian interpolant of $b_{|_{e^{(r)}}}$. Consequently, the interpolation inequality can be applied as follows (see [19, 2]),

$$\forall z \in e^{(r)}, \quad \|\mathrm{D}^p(b(z) - z)\| \leq ch^{r+1-p}, \qquad \text{for any } 0 \leq p \leq r+1.$$

This interpolation result combined with (28) is replaced in (33) to obtain,

$$\begin{aligned}
\|\mathrm{D}^m_{\hat{x}}(b(y) - y)\|_{\mathrm{L}^\infty(\hat{T}\backslash\hat{\sigma})} &\leq c\sum_{p=1}^{m} \Big(h^{r+1-p} \sum_{i \in E(m,p)} \prod_{q=1}^{m} (\frac{h}{(\lambda^*)^q})^{i_q} \Big) \\
&\leq c\sum_{p=1}^{m} \Big(h^{r+1-p} \frac{h^{\sum_{q=1}^m i_q}}{(\lambda^*)^{\sum_{q=1}^m q i_q}} \Big) \leq c\sum_{p=1}^{m} \Big(h^{r+1-p} \frac{h^p}{(\lambda^*)^m} \Big) \leq c\frac{h^{r+1}}{(\lambda^*)^m},
\end{aligned}$$

where the constant $c > 0$ is independent of h. This concludes the proof. □

Now, we introduce the mapping $\rho_{T^{(r)}}$, such that $F^{(e)}_{T^{(r)}} = F^{(r)}_T + \rho_{T^{(r)}}$ transforms $\hat{T}$ into the exact triangle $T^{(e)}$.

Proposition 8. *Let $\rho_{T^{(r)}} : \hat{x} \in \hat{T} \mapsto \rho_{T^{(r)}}(\hat{x}) \in \mathbb{R}^d$, be given by,*

$$\rho_{T^{(r)}}(\hat{x}) := \begin{cases} 0 & \text{if } \hat{x} \in \hat{\sigma}, \\ (\lambda^*)^{r+2}(b(y) - y) & \text{if } \hat{x} \in \hat{T} \backslash \hat{\sigma}. \end{cases}$$

The mapping $\rho_{T^{(r)}}$ is of class $\mathcal{C}^{r+1}$ on $\hat{T}$ and there exist a constant $c > 0$ independent of h such that,

$$\|\mathrm{D}^m \rho_{T^{(r)}}\|_{\mathrm{L}^\infty(\hat{T})} \leq ch^{r+1}, \quad \text{for} \quad 0 \leq m \leq r+1. \tag{34}$$

Proof. The mapping $\rho_{T^{(r)}}$ is of class $\mathcal{C}^{r+1}(\hat{T}\backslash\hat{\sigma})$, being the product of equally regular functions. Consider $0 \leq m \leq r+1$. Applying the Leibniz formula, we have,

$$\begin{aligned} \mathrm{D}^m \rho_{T^{(r)}}{}_{|_{\hat{T}\backslash\hat{\sigma}}} &= \mathrm{D}^m ((\lambda^*)^{r+2}(b(y) - y)) \\ &= \sum_{i=0}^{m} \tbinom{m}{i} (r+2)....(r+3-i)(\lambda^*)^{r+2-i} \mathrm{D}^{m-i}(b(y) - y). \end{aligned}$$

Then applying (32), we get, for $\hat{x} \in \hat{T}\backslash\hat{\sigma}$,

$$\|\mathrm{D}^m \rho_{T^{(r)}}(\hat{x})\| \leq c \sum_{i=0}^{m} (\lambda^*)^{r+2-i} \frac{ch^{r+1}}{(\lambda^*)^{m-i}} \leq ch^{r+1}(\lambda^*)^{r+2-m}.$$

Since $r + 2 - m > 0$, $(\lambda^*)^{r+2-m} \underset{\hat{x}\to\hat{\sigma}}{\longrightarrow} 0$. Consequently, $\mathrm{D}^m \rho_{T^{(r)}}$ can be continuously extended by 0 on $\hat{\sigma}$ when $0 \leq m \leq r+1$. Thus $\rho_{T^{(r)}} \in \mathcal{C}^{r+1}$ and the latter inequality ensures (34). □

We can now prove Proposition 3, as mentioned before, its proof relies on the previous propositions.

Proof of Proposition 3. Let $T^{(r)} \in \mathcal{T}^{(r)}_h$ be a non-internal curved element. Let $x = F^{(r)}_T(\hat{x}) \in T^{(r)}$ where $\hat{x} \in \hat{T}$. Following the equation (4), we recall that, $F^{(e)}_{T^{(r)}}(\hat{x}) = x + \rho_{T^{(r)}}(\hat{x})$. Then $G^{(r)}_h$ can be written as follows,

$$G^{(r)}_h{}_{|_{T^{(r)}}} = F^{(e)}_{T^{(r)}} \circ (F^{(r)}_T)^{-1} = (F^{(r)}_T + \rho_{T^{(r)}}) \circ (F^{(r)}_T)^{-1} = id_{|_{T^{(r)}}} + \rho_{T^{(r)}} \circ (F^{(r)}_T)^{-1}.$$

Firstly, with Proposition 8, $\rho_{T^{(r)}}$ is of class $\mathcal{C}^{r+1}(\hat{T})$ and $F^{(r)}_T$ is a polynomial, then $G^{(r)}_h$ is also $\mathcal{C}^{r+1}(T^{(r)})$.

Secondly, $F^{(r)}_T$ is a $\mathcal{C}^1$-diffeomorphism and there exists a constant $c > 0$ independent of h such that (see [10, page 239]),

$$\|\mathrm{D}(F^{(r)}_T)^{-1}\| \leq \frac{c}{h}. \tag{35}$$

Additionally, by applying (34) and (35), the following inequality holds,

$$\|\mathrm{D}(\rho_{T^{(r)}})\|_{\mathrm{L}^\infty(\hat{T})} \|\mathrm{D}((F^{(r)}_T)^{-1})\|_{\mathrm{L}^\infty(T^{(r)})} \leq ch^{r+1}\frac{c}{h} = ch^r < 1. \tag{36}$$

Then by applying [10, theorem 3], $F^{(r)}_T + \rho_{T^{(r)}}$ is a $\mathcal{C}^1$-diffeomorphism, being the sum of a $\mathcal{C}^1$-diffeomorphism and a $\mathcal{C}^1$ mapping, which satisfy (36). Therefore, $G^{(r)}_h = (F^{(r)}_T + \rho_{T^{(r)}}) \circ (F^{(r)}_T)^{-1}$ is a $\mathcal{C}^1$-diffeomorphism.

To obtain the first inequality of (5), we differentiate the latter expression,

$$\mathrm{D}G_h^{(r)}{}_{|_{T^{(r)}}} - \mathrm{Id}_{|_{T^{(r)}}} = \mathrm{D}(\rho_{T^{(r)}} \circ (F_T^{(r)})^{-1}) = \mathrm{D}(\rho_{T^{(r)}}) \circ ((F_T^{(r)})^{-1})\mathrm{D}(F_T^{(r)})^{-1}.$$

Using (34) and (35), we obtain,

$$\|\mathrm{D}G_h^{(r)}{}_{|_{T^{(r)}}} - \mathrm{Id}_{|_{T^{(r)}}}\|_{\mathrm{L}^\infty(T^{(r)})} \leq \|\mathrm{D}(\rho_{T^{(r)}})\|_{\mathrm{L}^\infty(\hat{T})}\|\mathrm{D}((F_T^{(r)})^{-1})\|_{\mathrm{L}^\infty(T^{(r)})} \leq ch^r,$$

where the constant $c > 0$ is independent of h. Lastly, the second inequality of (5) comes as a consequence of the first one, by definition of a Jacobian. □

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